

Minimal Matrix Representations of Decomposable Lie Algebras of Dimension Less Than or Equal to Five

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Presented by Consuelo Martínez

Received October 27, 2017

Abstract: We obtain minimal dimension matrix representations for each decomposable five-dimensional Lie algebra over \mathbb{R} and justify in each case that they are minimal.

Key words: Lie algebra, Lie group, minimal representation.

AMS Subject Class. (2000): 17Bxx, 53Cxx.

1. INTRODUCTION

Given a real Lie algebra \mathfrak{g} of dimension n a well known theorem due to Ado asserts that \mathfrak{g} has a faithful representation as a subalgebra of $\mathfrak{gl}(p, \mathbb{R})$ for some p . In several recently published papers the authors and others have investigated the problem of finding minimal dimensional representations of indecomposable Lie algebras of dimension five and less [7, 5, 2, 3]. In particular, in [3] minimal representations have been found for all the *indecomposable* algebras of dimension five and less. There remains the question of finding minimal dimensional representations of the *decomposable* Lie algebras of dimension five and less and that is the goal of this paper. It should be regarded as a sequel to [3] and as a result, we now have minimal dimensional representations of *all* Lie algebras of dimension up to and including five.

Burde [1] has defined an invariant $\mu(\mathfrak{g})$ of a Lie algebra \mathfrak{g} to be the dimension of its minimal faithful representation. In case an algebra \mathfrak{g} is decomposable, that is, $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ and one has representations for \mathfrak{g}_1 and \mathfrak{g}_2 , then there is an obvious way to construct a representation for \mathfrak{g} , that is, by taking a block diagonal representation, so that $\mu(\mathfrak{g}_1 \oplus \mathfrak{g}_2) \leq \mu(\mathfrak{g}_1) + \mu(\mathfrak{g}_2)$. However, it is an interesting phenomenon that in some cases there are representations of smaller

size than this diagonal representation, so that $\mu(\mathfrak{g}_1 \oplus \mathfrak{g}_2) < \mu(\mathfrak{g}_1) + \mu(\mathfrak{g}_2)$ is possible. A particular circumstance in which this latter inequality holds is when $\mathfrak{g}_1 = \mathbb{R}$, as an abelian one-dimensional Lie algebra and \mathfrak{g}_2 is indecomposable: then $\mu(\mathfrak{g}_1 \oplus \mathfrak{g}_2) = \mu(\mathfrak{g}_2)$ and the representation for $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ is obtained by adding multiples of the identity to the minimal dimensional representation for \mathfrak{g}_2 . The principal concern of the present paper is to furnish examples where $\mu(\mathfrak{g}_1 \oplus \mathfrak{g}_2) < \mu(\mathfrak{g}_1) + \mu(\mathfrak{g}_2)$ but that are not obtained by the trivial mechanism of adding multiples of the identity.

In this paper we shall use the classification of the low-dimensional *indecomposable* Lie algebras found in [6] which is in turn taken from [4]. Such algebras are denoted as $A_{m,n}$ where m denotes the dimension of the algebra and n the n th one in the list. We shall need indecomposable algebras of dimension from 2 to 4. In dimension 2, the algebra $A_{2,1}$ is the unique non-abelian algebra. In dimension 3 there are two simple algebras $A_{3,8} = \mathfrak{sl}(2, \mathbb{R})$ and $A_{3,9} = \mathfrak{so}(3)$ and the remaining seven are solvable. In dimension 4 there are no simple or semi-simple algebras and all 12 classes of algebra are solvable.

An outline of this paper is as follows. In Section 2 we list all the *decomposable* Lie algebras of dimension ≤ 5 . In Section 3 we consider classes of algebras given in Section 2, for example abelian Lie algebras. To show that μ has a certain value we argue in every case that μ has a certain lower bound; then all that is required is to exhibit a particular representation for which that value of μ is attained. In Section 4 we consider algebras of the form $A_{2,1} \oplus A_{3,n}$, which need slightly lengthier arguments. Finally, in Section 5 we give a list of representations with at least one for each decomposable Lie algebra where the value of μ is attained. Occasionally we give more than one such representation if it seems to be of particular interest although we do not consider the difficult issue of the equivalence of different representations.

2. CLASSIFICATION OF DECOMPOSABLE LIE ALGEBRAS

The first step is to determine, as *abstract Lie algebras*, all the *decomposable* Lie algebras of dimension ≤ 5 . Here is a summary of all such algebras.

- dimension 2: abelian \mathbb{R}^2 .
- dimension 3: abelian \mathbb{R}^3 ; nonabelian $\mathbb{R} \oplus A_{2,1}$.
- dimension 4: abelian \mathbb{R}^4 ; nonabelian: $\mathbb{R}^2 \oplus A_{2,1}$; $A_{2,1} \oplus A_{2,1}$; $\mathbb{R} \oplus A_{3,n}$.
- dimension 5: abelian \mathbb{R}^5 ; nonabelian $\mathbb{R}^3 \oplus A_{2,1}$; $\mathbb{R} \oplus A_{2,1} \oplus A_{2,1}$; $\mathbb{R}^2 \oplus A_{3,n}$; $A_{2,1} \oplus A_{3,n}$; $\mathbb{R} \oplus A_{4,n}$.

3. MINIMAL DIMENSION MATRIX REPRESENTATIONS

Now we shall consider each of the algebras above with regard to finding minimal dimension matrix representations.

3.1. ABELIAN SUBALGEBRAS. First of all for an abelian Lie algebra of dimension n we have $\mu = \lceil 2\sqrt{n-1} \rceil$ [1]. For $n = 2, 3, 4, 5$ we obtain $\mu(\mathbb{R}^n) = 2, 3, 4, 4$. Representations will be given in Section 5.

3.2. ALGEBRAS THAT HAVE \mathbb{R} AS A SINGLE SUMMAND. In each of the cases $\mathbb{R} \oplus A_{2,1}, \mathbb{R} \oplus A_{3,1}, \dots, \mathbb{R} \oplus A_{3,7}, \mathbb{R} \oplus A_{4,1}, \dots, \mathbb{R} \oplus A_{4,12}$ the second factor is solvable. Lie's Theorem guarantees an upper triangular representation for the adjoint representation, at least when we complexify the algebra. In fact for all these algebras representations have been found which are almost upper triangular and of minimal dimension [2, 3]. By "almost upper triangular" we mean that the complexified algebra is upper triangular. As such, we can simply add multiples of the identity so as to accommodate the extra factor of \mathbb{R} . In the case of $\mathbb{R} \oplus A_{2,1}$, equivalently we are simply looking at the space of 2×2 upper triangular matrices. In the case of $\mathbb{R} \oplus A_{3,8}$ we have $\mathbb{R} \oplus \mathfrak{sl}(2, \mathbb{R})$, which is isomorphic to $\mathfrak{gl}(2, \mathbb{R})$ and therefore $\mu = 2$. In the case of $\mathbb{R} \oplus A_{3,9}$ we have $\mathbb{R} \oplus \mathfrak{so}(3)$ and again we can simply add multiples of the identity to the standard three-dimensional representation of $\mathfrak{so}(3)$. See Section 5 for such representations.

3.3. DIMENSION FOUR NON-ABELIAN. Consulting the list in Section 2, the first classes of algebra for which we have not yet supplied a representation are $\mathbb{R}^2 \oplus A_{2,1}$ and $A_{2,1} \oplus A_{2,1}$. In the first case there is an abelian three-dimensional subalgebra and consequently $\mu \geq 3$; in fact $\mu = 3$ and we give two representations in Section 5. For $A_{2,1} \oplus A_{2,1}$ clearly $\mu > 2$ because the only four-dimensional subalgebra with $\mu = 2$ is $\mathfrak{gl}(2, \mathbb{R})$; however, we can find representations in $\mathfrak{gl}(3, \mathbb{R})$.

3.4. DIMENSION FIVE NON-ABELIAN.

3.4.1. $\mathbb{R}^3 \oplus A_{2,1}$. The algebra has a four dimensional abelian subalgebra and as such $\mu \geq 4$. In fact $\mu = 4$.

3.4.2. $\mathbb{R} \oplus A_{2,1} \oplus A_{2,1}$. For this algebra $\mu = 3$.

3.4.3. $\mathbb{R}^2 \oplus A_{3,n}$. For $1 \leq n \leq 7$ each of the algebras has a four dimensional abelian subalgebra and as such $\mu \geq 4$. In fact in each of these cases we can simply take a block diagonal representation of each three dimensional representation $\mathbb{R} \oplus A_{3,n}$ with \mathbb{R} giving $\mu = 4$. For $\mathbb{R}^2 \oplus A_{3,8}$ we can again take a block diagonal representation of $\mathbb{R} \oplus A_{3,8} \approx \mathfrak{gl}(2, \mathbb{R})$ with \mathbb{R} giving $\mu = 3$. Finally for $\mathbb{R}^2 \oplus A_{3,9}$ the fact that $\mu \geq 4$ follows from an application of Schur's Lemma. Again we can take a block diagonal representation.

4. ALGEBRAS $A_{2,1} \oplus A_{3,n}$

It remains to discuss the cases $A_{2,1} \oplus A_{3,n}$ where $1 \leq n \leq 9$. For $1 \leq n \leq 7$ these algebras have a three dimensional abelian subalgebra and as such $\mu \geq 3$. On the other hand for each $A_{3,n}$ we have $\mu \leq 3$. Hence for $A_{2,1} \oplus A_{3,n}$ we must have $\mu \leq 5$ by taking a block diagonal representation.

LEMMA 4.1. *Over \mathbb{R} every 3×3 matrix is equivalent under change of basis to one of the following:*

$$a) \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{bmatrix}, \quad b) \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{bmatrix}, \quad c) \begin{bmatrix} \alpha & 1 & 0 \\ -1 & \alpha & 0 \\ 0 & 0 & \beta \end{bmatrix}, \quad d) \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}.$$

LEMMA 4.2. *Any abelian three-dimensional subalgebra of $\mathfrak{gl}(3, \mathbb{R})$ contains a multiple of the identity.*

Proof. We refer to Lemma 4.1. In case d) the space of matrices that commutes with the given matrix is of the form $\begin{bmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{bmatrix}$. Likewise in case c) the space of matrices that commutes with the given matrix is of the form $\begin{bmatrix} a & b & 0 \\ -b & a & 0 \\ 0 & 0 & c \end{bmatrix}$ and hence the result follows in these two cases. In case b) if $\lambda \neq \mu$ the centralizer consists of matrices of the form $\begin{bmatrix} a & b & c \\ 0 & a & 0 \\ 0 & 0 & c \end{bmatrix}$ so if there is an abelian three-dimensional algebra it will contain I . In case b) if $\lambda = \mu$ the centralizer consists of matrices of the form $\begin{bmatrix} a & b & c \\ 0 & a & 0 \\ 0 & d & e \end{bmatrix}$. Now we may assume that $e = a$ or else we would be in a different subcase. Given two such commuting matrices by subtracting multiples of the original matrix we may assume that $b = 0$ and so a linear combination of them will be a multiple of I .

Finally, in case a) we can assume that all three matrices in the algebra have real eigenvalues of algebraic multiplicity one; otherwise the algebra would fall into case b), c) or d). Thus all three matrices are diagonalizable and

since they commute, simultaneously diagonalizable and linearly independent. Hence again the algebra will contain I . ■

COROLLARY 4.3. *A subalgebra of $\mathfrak{gl}(3, \mathbb{R})$ cannot have a subalgebra isomorphic to $A_{2,1} \oplus A_{3,n}$ where $1 \leq n \leq 7$.*

COROLLARY 4.4. *A subalgebra of $\mathfrak{gl}(3, \mathbb{R})$ that contains an abelian three-dimensional subalgebra is decomposable as an abstract Lie algebra.*

In view of Corollary 4.3 it follows for $A_{2,1} \oplus A_{3,n}$, where $1 \leq n \leq 7$, that $\mu \geq 4$. In fact $\mu = 4$.

It remains to discuss $A_{2,1} \oplus A_{3,8}$ and $A_{2,1} \oplus A_{3,9}$. In the first case we cannot have $\mu = 3$. If we could then the representation for $A_{3,8} = \mathfrak{sl}(2, \mathbb{R})$ would be either the 3×3 irreducible representation, which is excluded because the only matrices that commute with it are multiples of the identity and so $A_{2,1}$ could not be accommodated; or, the 2×2 irreducible representation augmented by an extra column of rows and zeros. However, in this latter case the centralizer of $\mathfrak{sl}(2, \mathbb{R})$ is two-dimensional abelian and again there would not be room for $A_{2,1}$. The block diagonal representation of $A_{2,1}$ and $A_{3,8}$ is four-dimensional and so $\mu = 4$.

In the case of $A_{2,1} \oplus A_{3,9}$ again $\mu \neq 3$ because of Schur's Lemma. We can apply Schur's Lemma again to show that $\mu \neq 4$. In fact one needs to exercise some caution because of the same reason that the 4×4 standard representation of $\mathfrak{so}(4) \approx \mathfrak{so}(3) \oplus \mathfrak{so}(3)$; as such one can find alternative 4×4 representations for $\mathbb{R}^2 \oplus A_{3,9}$ but not for $A_{2,1} \oplus A_{3,9}$. On the other hand the block diagonal representation of $A_{2,1}$ and $A_{3,9}$ is five-dimensional and so $\mu = 5$.

5. THE REPRESENTATIONS

5.1. DIMENSION THREE.

5.1.1. \mathbb{R}^3 .

$$\begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix} \begin{bmatrix} x & y & z \\ 0 & x & 0 \\ 0 & 0 & x \end{bmatrix}.$$

5.1.2. $\mathbb{R} \oplus A_{2,1}$.

$$\begin{bmatrix} x & z \\ 0 & y \end{bmatrix}.$$

5.2. DIMENSION FOUR.

5.2.1. \mathbb{R}^4 .

$$\begin{bmatrix} w & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & y & 0 \\ 0 & 0 & 0 & z \end{bmatrix} \begin{bmatrix} 0 & 0 & w & x \\ 0 & 0 & y & z \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} w & 0 & x & z \\ 0 & w & 0 & y \\ 0 & 0 & w & 0 \\ 0 & 0 & 0 & w \end{bmatrix}.$$

5.2.2. $\mathbb{R}^2 \oplus A_{2,1} : [e_1, e_2] = e_2$.

$$\begin{bmatrix} x & w & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix} \begin{bmatrix} \alpha x + z & y & w \\ 0 & (\alpha + 1)x + z & 0 \\ 0 & 0 & \alpha x + z \end{bmatrix} (\alpha \in \mathbb{R}).$$

5.2.3. $A_{2,1} \oplus A_{2,1} : [e_1, e_2] = e_2, [e_3, e_4] = e_4$.

$$\begin{bmatrix} \alpha x + \lambda z & y & w \\ 0 & (\alpha + 1)x + \lambda z & 0 \\ 0 & 0 & \alpha x + \mu z \end{bmatrix} (\alpha, \lambda, \mu \in \mathbb{R}, \lambda^2 + \mu^2 \neq 0).$$

5.2.4. $\mathbb{R} \oplus A_{3,k} (1 \leq k \leq 9)$. Lie brackets are the same as for $A_{3,k} (1 \leq k \leq 9)$: use the representations given in [3] and add multiples of the identity.

5.3. DIMENSION FIVE.

5.3.1. \mathbb{R}^5 .

$$\begin{bmatrix} q & 0 & w & x \\ 0 & q & y & z \\ 0 & 0 & q & 0 \\ 0 & 0 & 0 & q \end{bmatrix}.$$

5.3.2. $\mathbb{R}^3 \oplus A_{2,1} : [e_1, e_2] = e_2$.

$$\begin{bmatrix} x & y & 0 & 0 \\ 0 & z & 0 & 0 \\ 0 & 0 & w & 0 \\ 0 & 0 & 0 & q \end{bmatrix}.$$

$$5.3.3. \mathbb{R} \oplus A_{2,1} \oplus A_{2,1} : [e_2, e_3] = e_2, [e_4, e_5] = e_4.$$

$$\begin{bmatrix} x & q & w \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}.$$

$$5.3.4. \mathbb{R} \oplus A_{4.8,4.9b(-1 < b \leq 1)} : [e_2, e_3] = e_1, [e_1, e_4] = (b+1)e_1, [e_2, e_4] = e_2, [e_3, e_4] = be_3.$$

$$\begin{bmatrix} q + (b+1)w & x & z \\ 0 & q + bw & y \\ 0 & 0 & q \end{bmatrix}.$$

$$5.3.5. \mathbb{R} \oplus A_{4.12} : [e_1, e_3] = e_1, [e_2, e_3] = e_2, [e_1, e_4] = -e_2, [e_2, e_4] = e_1.$$

$$\begin{bmatrix} z & w & x \\ -w & z & y \\ 0 & 0 & q \end{bmatrix}.$$

$$5.3.6. \mathbb{R}^2 \oplus A_{3,1} : [e_2, e_3] = e_1.$$

$$\begin{bmatrix} q & 0 & 0 & 0 \\ 0 & \alpha x + \beta y & x & z \\ 0 & 0 & w + \alpha x + \beta y & y \\ 0 & 0 & 0 & w + \alpha x + \beta y \end{bmatrix} (\alpha, \beta \in \mathbb{R}).$$

$$5.3.7. \mathbb{R}^2 \oplus A_{3,2} : [e_1, e_3] = e_1, [e_2, e_3] = e_1 + e_2.$$

$$\begin{bmatrix} q & 0 & 0 & 0 \\ 0 & w + (\alpha + 1)z & z & x \\ 0 & 0 & w + (\alpha + 1)z & y \\ 0 & 0 & 0 & w + \alpha z \end{bmatrix} (\alpha \in \mathbb{R}).$$

$$5.3.8. \mathbb{R}^2 \oplus A_{3,3}, \mathbb{R}^2 \oplus A_{3,4}, \mathbb{R}^2 \oplus A_{3.5a} : [e_1, e_3] = e_1, [e_2, e_3] = ae_2.$$

$$\begin{bmatrix} q & 0 & 0 & 0 \\ 0 & w + (b+1)z & 0 & x \\ 0 & 0 & w + (a+b)z & y \\ 0 & 0 & 0 & w + bz \end{bmatrix}.$$

$$5.3.9. \mathbb{R}^2 \oplus A_{3,6}, \mathbb{R}^2 \oplus A_{3,7a} : [e_1, e_3] = ae_1 - e_2, [e_2, e_3] = e_1 + ae_2.$$

$$\begin{bmatrix} q & 0 & 0 & 0 \\ 0 & w + (a + \beta)z & z & x \\ 0 & -z & w + (a + \beta)z & y \\ 0 & 0 & 0 & w + \beta z \end{bmatrix} (\beta \in \mathbb{R}).$$

$$5.3.10. \mathbb{R}^2 \oplus A_{3,8} : [e_1, e_2] = 2e_2, [e_1, e_3] = -2e_3, [e_2, e_3] = e_1.$$

$$\begin{bmatrix} q & 0 & 0 \\ 0 & w & x \\ 0 & y & z \end{bmatrix}.$$

$$5.3.11. \mathbb{R}^2 \oplus A_{3,9} : [e_1, e_2] = e_3, [e_2, e_3] = e_1, [e_3, e_1] = e_2.$$

$$\begin{bmatrix} q & 0 & 0 & 0 \\ 0 & w & z & -y \\ 0 & -z & w & x \\ 0 & y & -x & w \end{bmatrix}.$$

$$5.3.12. A_{2,1} \oplus A_{3,1} : [e_1, e_2] = e_2, [e_3, e_4] = e_5.$$

$$\begin{bmatrix} q & 0 & 0 & w \\ 0 & 0 & x & z \\ 0 & 0 & 0 & y \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$5.3.13. A_{2,1} \oplus A_{3,2} : [e_1, e_2] = e_2, [e_3, e_5] = e_3 + e_4, [e_4, e_5] = e_4.$$

$$\begin{bmatrix} q & 0 & 0 & w \\ 0 & z & z & x \\ 0 & 0 & z & y \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$5.3.14. A_{2,1} \oplus A_{3,3}, A_{2,1} \oplus A_{3,4}, A_{2,1} \oplus A_{3,5a} : [e_1, e_2] = e_2, [e_3, e_5] = e_3, [e_4, e_5] = ae_4.$$

$$\begin{bmatrix} q & 0 & 0 & w \\ 0 & az & 0 & x \\ 0 & 0 & z & y \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

5.3.15. $A_{2,1} \oplus A_{3,6}$, $A_{2,1} \oplus A_{3,7a} : [e_1, e_2] = e_2$, $[e_3, e_5] = ae_3 - e_4$, $[e_4, e_5] = e_3 + ae_4$.

$$\begin{bmatrix} q & 0 & 0 & w \\ 0 & az & z & x \\ 0 & -z & az & y \\ 0 & 0 & 0 & w + \beta z \end{bmatrix}.$$

5.3.16. $A_{2,1} \oplus A_{3,8} : [e_1, e_2] = e_2$, $[e_3, e_4] = 2e_4$, $[e_3, e_5] = -2e_5$, $[e_4, e_5] = e_3$. Use the block diagonal representation.

5.3.17. $A_{2,1} \oplus A_{3,9} : [e_1, e_2] = e_2$, $[e_3, e_4] = e_5$, $[e_4, e_5] = e_3$, $[e_5, e_3] = e_4$. Use the block diagonal representation.

ACKNOWLEDGEMENTS

The authors thank the Qatar Foundation and Virginia Commonwealth University in Qatar for funding this project.

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